FURTHER PROPERTIES OF THE ENERGY-MOMENTUM COMPLEX IN GENERAL RELATIVITY

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CONTENTS

Synopsis

It is shown that the energy-momentum complex $\mathcal{T}_i^k$ introduced by MØLLER into the theory of general relativity is uniquely determined, when taken as a function of the metric tensor and its derivatives of the first and second orders, by two transformation requirements: 1) $\mathcal{T}_i^k$ is an affine tensor density (of weight one) so that the total energy and momentum of a closed system are transformed as a vector in linear (affine) transformations, just like the energy and momentum of a free particle; 2) $\mathcal{T}_4^4$ is a scalar density in arbitrary spatial transformations so that the total energy in a volume of space is independent of the system of spatial coordinates used. Further it is shown that in empty space it is possible, in accordance with the principle of equivalence, to introduce coordinates along a geodesic such that the gravitational energy-momentum complex vanishes along the geodesic.

1. Introduction

When EINSTEIN introduced the law of conservation of energy and momentum into the theory of general relativity[1], several objections were raised against it. These arose from the fact that the energy-momentum components t_i^k of the gravitational field did not form a tensor, whereas the components T_i^k of matter did. It was, e. g., shown by BAUER that for an inertial system in which no matter was present, i. e. no gravitational field, the introduction of polar instead of Cartesian space coordinates into the metric of special relativity led to components t_i^k different from zero. In particular, the total energy turned out to be infinite. LEVI-CIVITA and LORENTZ proposed an alternative expression for the energy-momentum components of the gravitational field, viz. the tensor $\frac{1}{\varkappa} G_i^k$, where $G_i^k = R_i^k - \frac{1}{2} \delta_i^k R$. This proposal was rejected by EINSTEIN on the grounds that, since $T_i^k + \frac{1}{\varkappa} G_i^k = 0$ always and everywhere, according to the field equations, the total energy of a system is zero from the start, and therefore this law of conservation does not require the continued existence of the system. A material system can disintegrate into nothing without leaving any trace[2].

Finally EINSTEIN[3] showed that his formulation of the law of conservation of energy and momentum led to an unambiguous and satisfactory definition of the total energy and momentum of a closed system, independent of the choice of coordinates inside a surface surrounding the system. However, no unambiguous definition could be given of the energy or momentum of a part of a closed system. Therefore it was generally accepted that the localization of energy and momentum had no meaning in the theory of general relativity.

In recent papers, MØLLER[4, 5, 6] has derived and discussed extensively a new energy-momentum complex in general relativity. (The term "complex" is used, as by LORENTZ, to denote a quantity which is not transformed as a tensor in arbitrary space-time transformations.) In the first paper, "On the Localization of the Energy of a Physical System in the

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General Theory of Relativity", it was pointed out that the 44-component of Einstein's energy-momentum complex was not transformed as a scalar density even in purely spatial transformations. It was therefore not suitable for defining an energy density. This was the reason for the above-mentioned absurd result derived by BAUER. Using the fact that the energy-momentum complex is not uniquely determined by the requirement that its ordinary divergence vanishes, MØLLER succeeded in deriving a complex T_i^k having all the satisfactory features of Einstein's energy-momentum complex, but such that T_4^4 and $T_4^\varkappa$ ($\varkappa = 1, 2, 3$) behaved like scalar and 3-vector densities, respectively, in arbitrary spatial transformations. With this new expression for T_i^k, the energy of a part of a closed system is invariant in spatial transformations.

As was shown by MØLLER[6, 7], T_i^k can be derived from a variational principle, where the quantity to be varied is the curvature scalar density $\mathfrak{R} = \sqrt{-g}\, R$. However, it is possible to define in the theory of general relativity, as in any generally covariant theory where the field equations can be derived from a variational principle, an infinite number of quantities which satisfy conservation laws[8]. It is desirable to select among these a unique, physically significant quantity describing the energy and momentum of the field. The question of the uniqueness of Møller's energy-momentum complex T_i^k has been considered by himself in another paper[5], where he shows that T_i^k is determined uniquely by the following three conditions:

1) T_i^k is an affine tensor density.
2) T_4^4, $T_4^\varkappa$ ($\varkappa = 1, 2, 3$) are scalar and 3-vector densities in arbitrary spatial transformations.
3) The superpotential χ_i^{kl} from which T_i^k is derived, $T_i^k = \chi_{i,l}^{kl}\,{}^*$, depends on first-order derivatives of the metric tensor up to the second degree, and does not contain higher derivatives.

T_i^k may be separated into a matter part, $\sqrt{-g}\, T_i^k$, and a gravitational part, $\sqrt{-g}\, t_i^k$. According to the principle of equivalence it should be possible to eliminate the gravitational field, and thus make t_i^k vanish, at any point by a suitable choice of coordinate systems. Since t_i^k depends on the second derivatives of the metric tensor, it will not vanish in all coordinate systems which are geodesic, i. e. in which $g_{ik,l} = 0$. MØLLER has shown, however, that t_i^k can be made to vanish at any point where no matter is present, in a wide class of geodesic coordinate systems, viz. those which are "locally normal" at the point[6].

* A comma denotes partial and a semicolon covariant differentiation.

The present paper falls into two main parts. The first part deals with the uniqueness of Møller's energy-momentum complex. After considering conservation laws and the transformation properties of the energy-momentum complex we show that the complex T_i^k, satisfying a conservation law and depending on the metric tensor and its derivatives of the first and second orders, is uniquely determined by conditions 1) and 2) above, i. e. that T_i^k is an affine tensor density and T_4^4, $T_4^\varkappa$ are scalar and vector densities in spatial transformations. The restriction on the degree of the first-order derivatives of g_{ik} in χ_i^{kl} can be dropped. Derivatives of g_{ik} higher than the second are excluded from T_i^k, and therefore derivatives higher than the first from χ_i^{kl}, since the field equations themselves are restricted to the second order. It is also easily seen that it is impossible to form a quantity T^{ik} such that T^{44} is a scalar density in spatial transformations.

In the second part it is shown that along a geodesic one can introduce coordinate systems such that, with no matter present, the gravitational energy-momentum complex t_i^k vanishes along the geodesic. This is an extension of Møller's result for a point where no matter is present. It shows that an observer falling freely in a gravitational field can introduce a system of coordinates such that the effects of the gravitational field are eliminated.

2. Conservation Laws and Transformation Properties

The conservation laws of energy and momentum are originally integral laws. For a closed system they state that a certain well-defined space integral over the system at a certain time, called its energy or momentum, remains constant in time:

$$\frac{d}{dt} \int T \, dx^1 \, dx^2 \, dx^3 = 0 \,. \tag{1}$$

For a part of a closed system the conservation laws state that the rate of decrease of, say, the energy in a space volume V at a certain time is equal to the flux of energy through the boundary surface S of V:

$$-\frac{d}{dt} \int_V T \, dx^1 \, dx^2 \, dx^3 = \int_S \mathfrak{F}^\varkappa \, dS_\varkappa \,. \tag{2}$$

It is, however, more convenient to have the conservation laws in differential form. The differential conservation law equivalent to (2) is, by Gauss' theorem,

$$\frac{\partial \mathcal{T}}{\partial t} + \frac{\partial \mathfrak{J}^{\varkappa}}{\partial x^{\varkappa}} = 0. \tag{3}$$

In special relativity the differential conservation laws for energy and momentum are

$$\frac{\partial T_i^k}{\partial x^k} = T_{i,\,k}^k = 0, \tag{4}$$

where T_i^k is the energy-momentum tensor of matter. The natural generalization of (4) in general relativity is obtained by equating the covariant divergence of the tensor T_i^k to zero:

$$T_{i;\,k}^k = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^k} (\sqrt{-g}\, T_i^k) - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} T^{kl} = 0. \tag{5}$$

As is well known, this equation does not lead to integral conservation laws of the form (1) or (2), i. e., there is no conservation law for matter alone. Only an equation of the form (4) is equivalent to integral conservation laws. It is possible, however, to bring (5) into the required form by means of the field equations. One then obtains the conservation laws for matter and gravitational field in a differential form:

$$\frac{\partial \mathcal{T}_i^k}{\partial x^k} = \mathcal{T}_{i,\,k}^k = 0, \tag{6}$$

where

$$\mathcal{T}_i^k = \sqrt{-g}\,(T_i^k + t_i^k). \tag{7}$$

Here T_i^k is the energy-momentum tensor of matter, and t_i^k refers to the gravitational field. Of course $\mathcal{T}_i^k$, and therefore t_i^k, are not uniquely determined in this way, for a quantity with a vanishing divergence can be added to $\mathcal{T}_i^k$. By means of the field equations the matter variables can be eliminated and $\mathcal{T}_i^k$ expressed solely in terms of the metric tensor and its derivatives. As noted in section 1, it is natural to exclude derivatives higher than the second from $\mathcal{T}_i^k$, but there is still a wide choice of expressions for $\mathcal{T}_i^k$, which it is desirable to restrict.

The principle of general relativity requires the validity of equation (6) in all systems of coordinates. This puts restrictions on the transformation properties of $\mathcal{T}_i^k$. It is clear, e. g. from eq. (5), that $\mathcal{T}_i^k$ cannot be a tensor density (of weight one) in arbitrary space-time transformations, but only in linear (affine) space-time transformations, i. e. it can be an affine ten-

sor density. Further it can be shown that in arbitrary spatial transformations, $x'^\iota = f^\iota(x^\varkappa)$, $x'^4 = x^4$, $(\iota, \varkappa = 1, 2, 3)$, T_4^4 and $T_4^\varkappa$ can be scalar and tensor densities, whereas T_ι^4 and $T_\iota^\varkappa$ cannot be vector and tensor densities.

Turning now to the question of determining physically reasonable transformation properties of T_i^k, we first consider the total energy and momentum of a closed system

$$P_i = \frac{1}{c} \int T_i^4 \, dx^1 \, dx^2 \, dx^3. \tag{8}$$

It is natural to require that this is transformed like the energy and momentum of a free particle, i. e. as a vector, in linear transformations. This means that T_i^k must be an affine tensor density (of weight one).

Now consider the gravitational energy in a small, or infinitesimal, region. It is clear that this energy will depend on the coordinate system used. According to the principle of equivalence it is possible to introduce a coordinate system in which the gravitational field vanishes. In such a system all the components of the gravitational energy-momentum complex, in particular the energy density, should vanish. The elimination of the gravitational field requires the introduction of an accelerated (freely falling) frame of reference, and hence the energy-momentum complex cannot be a tensor density in transformations to such a frame. The transformations involve time, but not linearly; so they are not affine. Thus it follows from the principle of equivalence that T_i^k cannot be a tensor density in arbitrary space-time transformations, whereas it can be an affine tensor density.

Within a given frame of reference, an arbitrary change in the spatial coordinates only will not eliminate or affect the gravitational field. It is then natural to require that the gravitational energy in a spatial region be invariant in arbitrary spatial transformations $x'^\iota = f^\iota(x^\varkappa)$, $x'^4 = x^4$, $(\iota, \varkappa = 1, 2, 3)$, which simply amount to a renaming of the points of reference (points with constant spatial coordinates) without any change of the rate or setting of the coordinate clocks. This is the case if T_4^4 behaves like a scalar density in such transformations. Further, if $T_4^\varkappa$ is a 3-vector density, the integrals in (2) with $T = T_4^4$ and $\mathfrak{F}^\varkappa = c T_4^\varkappa$ are invariant in spatial transformations. In that case one may talk of conservation of energy in any region of space within a given frame of reference, regardless of the system of spatial coordinates used.

The situation is different as regards the momentum. T_ι^4, $T_\iota^\varkappa$ cannot be vector and tensor densities in spatial transformations. Even if they were,

the corresponding integrals in equation (2) would not have simple trans-formation properties in such transformations (see reference 4, § 4).

Thus it is possible, within any given frame of reference, to give an unambiguous interpretation, i. e. one independent of the choice of spatial coordinates, of the conservation of the energy in any region of space, provided T_4^4 and $T_4^\varkappa$ have the above-mentioned properties, but not of the conservation of the momentum.

On the basis of these considerations we may set up the following trans-formation requirements for T_i^k, consistent with eq. (6) being valid in all coordinate systems:

1) T_i^k must be an affine tensor density;
2) T_4^4, $T_4^\varkappa$ must be scalar and vector densities in spatial transformations $x'^\iota = f^\iota(x^\varkappa)$, $x'^4 = x^4$, $(\iota, \varkappa = 1, 2, 3)$.

Now, equation (6) is satisfied identically in all coordinate systems if one writes

$$T_i^k = \chi_{i,\,l}^{kl},\tag{9}$$

where

$$\chi_i^{kl} = -\chi_i^{lk}.\tag{10}$$

With T_i^k restricted to second derivatives, χ_i^{kl} must be restricted to first-order derivatives of the metric tensor.

Then T_i^k will have the required transformation properties if

1) χ_i^{kl} is an affine tensor density,
2) $\chi_4^{4\lambda}$, $\chi_4^{\varkappa\lambda}$ are vector and tensor densities in spatial transformations $(\varkappa, \lambda = 1, 2, 3)$.

We shall now show that the superpotential χ_i^{kl} formed from the metric tensor and its first-order derivatives is uniquely determined by these trans-formation requirements. Hence T_i^k is uniquely determined by the corres-ponding requirements.

3. Spatially Covariant Expressions Containing First-Order Derivatives of the Metric Tensor

Consider the problem of forming a rational integral function of the metric tensor and its first-order derivatives which is covariant in the spatial transformation

$$x'^\iota = f^\iota(x^\varkappa), \quad x'^4 = x^4, \quad (\iota, \varkappa = 1, 2, 3).\tag{11}$$

The transformation coefficients for (11) are

$$\alpha_k^i = \frac{\partial x'^i}{\partial x^k} : \alpha_\varkappa^\iota = \frac{\partial x'^\iota}{\partial x^\varkappa}, \; \alpha_4^\iota = \alpha_\iota^4 = 0, \; \alpha_4^4 = 1,$$

$$\breve{\alpha}_k^i = \frac{\partial x^i}{\partial x'^k} : \breve{\alpha}_\varkappa^\iota = \frac{\partial x^\iota}{\partial x'^\varkappa}, \; \breve{\alpha}_4^\iota = \breve{\alpha}_\iota^4 = 0, \; \breve{\alpha}_4^4 = 1. \qquad (12)$$

The first-order derivatives of the metric tensor can be written in terms of the Christoffel symbols of the second kind and the metric tensor as follows:

$$g_{ij,k} = g_{im} \Gamma_{jk}^m + g_{jm} \Gamma_{ik}^m$$
$$g^{ij}{}_{,k} = -g^{im} \Gamma_{mk}^j - g^{jm} \Gamma_{mk}^i. \qquad (13)$$

Any expression containing the first-order derivatives can therefore be written in terms of the Christoffel symbols and the metric tensor. The problem is then to form a spatially covariant expression in terms of g_{ik}, g^{ik} and Γ_{kl}^i.

For an arbitrary space-time transformation the transformation law for Γ_{kl}^i is [9]

$$\Gamma'^i_{kl} = \alpha_r^i \breve{\alpha}_k^s \breve{\alpha}_l^t \Gamma_{st}^r + \alpha_r^i \breve{\alpha}_{k,l}^r. \qquad (14)$$

Thus Γ_{kl}^i is not a tensor in general, because of the second term on the right-hand side. This term vanishes when the transformations are linear, i. e. Γ_{kl}^i is an affine tensor. For the spatial transformation (11) it is easily seen that the extra term vanishes if one of the indices i, k, l is equal to 4. Therefore

$$\Gamma_{kl}^4, \quad \Gamma_{4l}^i = \Gamma_{l4}^i \qquad (15)$$

are tensors in spatial transformations. Any expression containing only these symbols (and the metric tensor) will be spatially covariant. This is not the case with a general expression in Γ_{kl}^i, but it is possible that some particular combination of Γ_{kl}^i will be spatially covariant. For that to occur the extra non-tensor terms in the transformation law for the expression, arising from the extra term in (13), must somehow be cancelled. We shall now show that this is impossible.

Consider first an expression linear in Γ_{kl}^i, e. g. with a term of the type

$$\Gamma_{kl}^i g^{mn} g_{rs}, \qquad (16)$$

where $i, k, l \neq 4$. The transformation law for this term is

$$\Gamma'^{i}_{kl}\, g'^{mn}\, g'_{rs} = \alpha^{i}_{t}\, \breve{\alpha}^{u}_{k}\, \breve{\alpha}^{v}_{l}\, \alpha^{m}_{w}\, \alpha^{n}_{x}\, \breve{\alpha}^{y}_{r}\, \breve{\alpha}^{z}_{s}\, \Gamma^{t}_{uv}\, g^{wx}\, g_{yz}$$
$$+ \alpha^{i}_{t}\, \breve{\alpha}^{t}_{k,\,l}\, \alpha^{m}_{w}\, \alpha^{n}_{x}\, \breve{\alpha}^{y}_{r}\, \breve{\alpha}^{z}_{s}\, g^{wx}\, g_{yz}. \tag{17}$$

All the terms are symmetric in the same pairs of indices, (k, l), (m, n) and (r, s), and in no others. The only nontrivial way to cancel the extra term is to subtract a corresponding expression where the indices in which the term is symmetric are interchanged. Since all the terms of (17) are symmetric in the same pairs of indices, they would all be cancelled by such a subtraction, and nothing would remain.

For an expression containing products of Γ^{i}_{kl} the extra, non-tensor term in (14) would lead to several extra terms in the transformation law, similar to that in (17). Since, however, all the terms would be symmetric in the same pairs of indices, and in those only, it would be impossible to cancel the extra non-tensor terms without cancelling all the others as well.

The only way to obtain a covariant expression is to make the extra term in (14) vanish. For an arbitrary spatial transformation (11) this means that only the Christoffel symbols given in (15) can occur in the expression.

4. The Uniqueness of the Superpotential χ^{kl}_{i}

The considerations in section 2 led to the following requirements for the transformation properties of the superpotential χ^{kl}_{i}, depending on the metric tensor and its first-order derivatives:

1) χ^{kl}_{i} must be an affine tensor density;
2) $\chi^{4\lambda}_{4}$, $\chi^{\varkappa\lambda}_{4}$ must be vector and tensor densities in arbitrary spatial transformations.

Since χ^{kl}_{i} is to be a density (of weight one), it can be written

$$\chi^{kl}_{i} = \sqrt{-g}\, X^{kl}_{i}, \tag{18}$$

where X^{kl}_{i} is an affine tensor and $X^{4\lambda}_{4}$ a vector in spatial transformations. Being an affine tensor, X^{kl}_{i} must be a rational integral function of the metric tensor and its first-order derivatives.

In a spatially covariant expression for X^{4l}_{4} there can only be one upper and one lower index equal to 4 since X^{4l}_{4} is associated with an affine tensor of rank three, X^{kl}_{i}.

A spatially covariant expression for X^{4l}_{4}, antisymmetric in 4 and l, must be formed of the following quantities, and these only:

$$\Gamma^4_{mn}, \ \Gamma^n_{4m}, \ \Gamma^4_{4m}, \ \Gamma^l_{4m}, \ g_{mn}, \ g_{4n}, \ g^{mn}, \ g^{4n}, \ g^{ln}, \ \delta^4_4, \ \delta^l_4, \tag{19}$$

m and n representing dummy indices. g^{4l} cannot occur since the expression is to be antisymmetric in 4 and l, and $\delta^n_m, \ \delta^4_m, \ \delta^n_4$ would simply mean replacing a dummy index by 4 or another dummy index.

It is not possible to form X^{4l}_4 from $g_{ik}, \ g^{ik}, \ \delta^k_i$ alone, for a quantity formed from these would always have the same number of upper and lower indices. Every term must therefore have one or more Γ^i_{kl}. Terms of the third or a higher degree in Γ^i_{kl} cannot occur as they would have three or more indices equal to 4. This excludes, according to section 3, terms of the third and higher degrees in the first-order derivatives. Terms of the second degree in Γ^i_{kl} cannot occur since it is impossible to form from a product of two Γ^i_{kl}, the metric tensor, and the Kronecker symbol, a quantity with one more index on top than at bottom. This excludes terms of the second degree in the first-order derivatives. The only remaining possibility is to have terms linear in the Γ^i_{kl}, i. e. terms of the first degree in the first-order derivatives.

To form the quantity

$$X^{4l}_4 = - X^{l4}_4 \tag{20}$$

from the quantities in (19) and so that it is linear in the Γ^i_{kl}, consider first the use of Γ^4_{mn} with m, n dummy indices. Interchange of 4 and l in accordance with (20) would give Γ^l_{mn}, which is not covariant. Hence the only Γ's which can occur are $\Gamma^n_{4m}, \ \Gamma^4_{4m}$ and Γ^l_{4m}. This, however, excludes g_{4n}, δ^4_4 and δ^l_4 since there can be only one lower index equal to 4. The quantities left to form X^{4l}_4 are then

$$\Gamma^n_{4m}, \ \Gamma^4_{4m}, \ \Gamma^l_{4m}, \ g_{mn}, \ g^{mn}, \ g^{4n}, \ g^{ln}. \tag{21}$$

The possible positions for 4 and l as upper indices are given by

$$\Gamma^n_{4m} \, g^{4r} \, g^{ls}, \ \Gamma^4_{4m} \, g^{ls}, \ \Gamma^l_{4m} \, g^{4r}. \tag{22}$$

Matching the dummy indices m, n, r, s in all possible ways, one finds three expressions:

$$\Gamma^4_{4m} \, g^{lm}, \ \Gamma^l_{4m} \, g^{4m}, \ \Gamma^m_{4m} \, g^{4l}. \tag{23}$$

The last one is symmetric in 4 and l and therefore cannot occur in X^{4l}_4. From the other two one can form only one antisymmetric quantity

$$X^{4l}_4 = a \, (\Gamma^l_{4m} \, g^{4m} - \Gamma^4_{4m} \, g^{lm}), \tag{24}$$

which is thus uniquely determined, apart from an arbitrary constant.

It is clear that the quantity

$$X_4^{\varkappa\lambda} = a \left(\Gamma_{4m}^{\lambda} \, g^{\varkappa m} - \Gamma_{4m}^{\varkappa} \, g^{\lambda m} \right) \tag{25}$$

is spatially covariant, as required.

Expressing the Christoffel symbols in terms of first-order derivatives of the metric tensor, one finds

$$X_4^{4l} = a \left(g_{4n,\,m} - g_{4m,\,n} \right) g^{4m} \, g^{ln}. \tag{26}$$

The superpotential χ_i^{kl} is then given by

$$\chi_i^{kl} = a \sqrt{-g} \left(\Gamma_{im}^{l} \, g^{km} - \Gamma_{im}^{k} \, g^{lm} \right) \tag{27 a}$$

or

$$\chi_i^{kl} = a \sqrt{-g} \left(g_{in,\,m} - g_{im,\,n} \right) g^{km} \, g^{ln}. \tag{27 b}$$

This is just the expression derived by MØLLER, and it is thus seen to be uniquely determined by the two transformation properties given at the beginning of this section. It follows from Møller's work that the constant a is given by

$$a = 1/\varkappa = c^4/8\,\pi k, \tag{28}$$

where k is the Newtonian gravitational constant.

It is now easily seen that it is impossible to form an energy-momentum complex T^{ik} such that T^{44} is a scalar density in spatial transformations. To do so one would put $T^{ik} = \chi_{,\,l}^{ikl}$, where $\chi^{ikl} = -\chi^{ilk}$ and χ^{44l} must be a vector density in spatial transformations. To form this latter quantity one would have to use

$$\Gamma_{mn}^4,\, g_{mn},\, g^{mn},\, g^{4m},\, g^{ln},\, g^{4l}. \tag{29}$$

It is not possible to form it from g_{mn} and g^{mn} alone, nor from products of two Γ_{kl}^{i} since there must be three free upper indices. Products of three or more Γ_{kl}^{i} would give too many 4's, so only a linear expression in Γ_{kl}^{i} remains. Matching indices in

$$\Gamma_{mn}^4 \, g^{4r} \, g^{ls},\quad \Gamma_{mn}^4 \, g^{4l}, \tag{30}$$

one finds the following expressions:

$$\Gamma_{mn}^4 \, g^{4l} \, g^{mn},\quad \Gamma_{mn}^4 \, g^{4m} \, g^{ln}. \tag{31}$$

The index 4 in Γ_{mn}^4 cannot be replaced by l in the process of forming

$\chi^{4l4} = -\chi^{44l}$ since Γ^l_{mn} is not spatially covariant. The expressions in (31) are symmetric in the remaining 4 and l so that an expression antisymmetric in 4 and l cannot be formed from them. Thus it is impossible to form the required quantity χ^{44l}.

5. The Energy-Momentum Complex of the Gravitational Field

The energy-momentum complex of matter and gravitational field, $\mathcal{T}^k_i$, can be expressed solely in terms of the metric tensor and its first- and second-order derivatives. By means of equations (9), (27) and (28) $\mathcal{T}^k_i$ may be written

$$\mathcal{T}^k_i = \left\{ \frac{\sqrt{-g}}{\varkappa} \left(\Gamma^l_{im} g^{km} - \Gamma^k_{im} g^{lm} \right) \right\}_{,l} \qquad (32\,a)$$

or

$$\mathcal{T}^k_i = \left\{ \frac{\sqrt{-g}}{\varkappa} \left(g_{in,\,m} - g_{im,\,n} \right) g^{km} g^{ln} \right\}_{,l}. \qquad (32\,b)$$

$\mathcal{T}^k_i$ can be split up into a matter part, $\sqrt{-g}\, T^k_i$, and a gravitational part, $\sqrt{-g}\, t^k_i$, as in equation (7),

$$\mathcal{T}^k_i = \sqrt{-g} \left(T^k_i + t^k_i \right). \qquad (7)$$

This is, however, rather artificial and arbitrary since $\mathcal{T}^k_i$ can be expressed in terms of the metric tensor and its derivatives alone, the matter variables being eliminated entirely from the expression. Further, T^k_i and t^k_i are not conserved separately in a general coordinate system; only their sum is conserved. In general one has from (6), (7) and (5)

$$\left(\sqrt{-g}\, t^k_i \right)_{,\,k} = - \left(\sqrt{-g}\, T^k_i \right)_{,\,k} = - \frac{\sqrt{-g}}{2}\, g_{kl,\,i}\, T^{kl}. \qquad (33)$$

It is possible to introduce at any given point a geodesic coordinate system such that $g_{ik,\,l} = 0$ and therefore also $\Gamma^i_{kl} = 0$ at the point. As was first shown by Fermi, it is also possible, for any open curve in space-time, to introduce coordinate systems such that $g_{ik,\,l} = 0$ at every point of the curve. At points where $g_{ik,\,l} = 0$, it is reasonable to talk of a matter part and a gravitational part of $\mathcal{T}^k_i$ since these are conserved separately at such points, i. e.

$$\left(\sqrt{-g}\, t^k_i \right)_{,\,k} = \left(\sqrt{-g}\, T^k_i \right)_{,\,k} = 0. \qquad (34)$$

Møller has shown[6] that t^k_i can be written

$$t_i^k = -\frac{1}{2\varkappa} R \delta_i^k + \hat{t}_i^k \tag{35}$$

where

$$\varkappa \hat{t}_i^k = (\Gamma_{il}^l)^{,\,k} - (\Gamma_{il}^k)^{,\,l} - \Gamma_{il}^k (g_{,\,m}^{lm} + g^{lm} \Gamma_{mn}^n) - \Gamma_{in}^m \Gamma_{lm}^k g^{ln} \tag{36}$$

with

$$(\Gamma_{kl}^i)^{,\,m} = (\Gamma_{kl}^i)_{,\,n} g^{nm}. \tag{37}$$

At the origin of a geodesic coordinate system, (36) is reduced to

$$\varkappa \hat{t}_i^k = (\Gamma_{il}^l)^{,\,k} - (\Gamma_{il}^k)^{,\,l}. \tag{38}$$

By means of (37) and the relation

$$\Gamma_{kl}^i = g^{ir} \Gamma_{r,\,kl} \tag{39}$$

(38) may be written

$$\varkappa \hat{t}_i^k = [(\Gamma_{n,\,il})_{,\,m} - (\Gamma_{m,\,il})_{,\,n}] \, g^{km} \, g^{ln}. \tag{40}$$

Since this expression depends on second-order derivatives of the metric tensor, t_i^k will in general not vanish at the origin of a geodesic system of coordinates.

According to the principle of equivalence it should be possible, however, to eliminate the effects of the gravitational field at a point by a suitable choice of coordinates. Møller has shown[6] that where no matter is present, i. e. where $R = 0$, t_i^k does vanish at the origin in a large class of geodesic coordinate systems, the so-called normal or Riemannian systems. The physical significance of normal coordinates has also been discussed by other authors, who point out their correspondence to Minkowskian coordinates of special relativity[10].

Møller suggested that it would be possible to introduce coordinates along a geodesic such that $\hat{t}_i^k = 0$ along it, i. e. $t_i^k = 0$ where no matter is present. This is physically reasonable, for it means that an observer falling freely in a gravitational field can introduce coordinates such that the effects of the gravitational field are approximately eliminated in his neighbourhood. It actually turns out to be possible.

In the appendix it is shown that for a geodesic in Riemannian space, V_4, there exist coordinate systems such that for every point of the geodesic

$$\Gamma_{kl}^i = 0, \quad i, k, l = 1, 2, 3, 4, \tag{41}$$

$$\underset{(\varkappa\lambda\mu)}{S} (\Gamma_{\varkappa\lambda}^i)_{,\,\mu} = \frac{1}{3} (\Gamma_{\varkappa\lambda,\,\mu}^i + \Gamma_{\mu\varkappa,\,\lambda}^i + \Gamma_{\lambda\mu,\,\varkappa}^i) = 0, \quad \varkappa, \lambda, \mu = 1, 2, 3. \tag{42}$$

The coordinates are called Fermi coordinates for a geodesic. The spatial coordinates x^1, x^2, x^3 are just Riemannian or normal coordinates in the hypersurface orthogonal to the geodesic since (see A16)

$$x^\mu = z t^\mu, \quad \mu = 1, 2, 3, \tag{43}$$

where t^μ is a vector at the geodesic in this surface and z is the arc length along a geodesic in the surface whose direction is specified by t^μ. The fourth coordinate x^4 is proportional to the arc length along the geodesic. For a time-like geodesic it can be taken as c times the proper time. This coordinate system is clearly time-orthogonal, and with the above choice of the fourth coordinate one finds that $g_{44} = -1$. Thus

$$g_{4\mu} = 0, \quad g_{44} = -1 \tag{44}$$

so that the metric is

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu - (dx^4)^2. \tag{45}$$

The Fermi coordinates are a special case of geodesic coordinates. From (41) it is seen that

$$g_{ik,l} = 0, \quad g^{ik}_{,l} = 0. \tag{46}$$

Equation (40) therefore holds in this system of coordinates. Since (41) and (46) hold at every point of the geodesic, it follows that

$$(\Gamma^i_{kl})_{,4} = (\Gamma_{i,kl})_{,4} = 0, \; g_{ik,l,4} = g^{ik}_{,l,4} = 0 \tag{47}$$

because x^4 is proportional to the arc length along the geodesic. From (44) one has

$$g^{4\mu} = 0 \text{ and } g^{44} = -1. \tag{48}$$

Using (44), (47) and (48), one finds from (40) that

$$\hat{\imath}^4_4 = \hat{\imath}^\varkappa_4 = \hat{\imath}^4_\iota = 0 \tag{49}$$

and

$$\varkappa \hat{\imath}^\varkappa_\iota = [(\Gamma_{\nu,\iota\lambda})_{,\mu} - (\Gamma_{\mu,\iota\lambda})_{,\nu}]\, g^{\varkappa\mu} g^{\lambda\nu}. \tag{50}$$

From the condition (42), satisfied by Fermi coordinates, it follows that

$$3 \underset{(\varkappa\lambda\mu)}{S} (\Gamma_{i,\varkappa\lambda})_{,\mu} = (\Gamma_{i,\varkappa\lambda})_{,\mu} + (\Gamma_{i,\mu\varkappa})_{,\lambda} + (\Gamma_{i,\lambda\mu})_{,\varkappa} = 0. \tag{51}$$

Putting $i = \iota$, one has

$$(\Gamma_{\iota,\varkappa\lambda})_{,\mu} + (\Gamma_{\iota,\mu\varkappa})_{,\lambda} + (\Gamma_{\iota,\lambda\mu})_{,\varkappa} = 0. \tag{52}$$

From (41) it is seen that

$$\Gamma_{\iota,\,\varkappa\lambda} = 0.\tag{53}$$

Equations (53) and (52) are the very equations satisfied by Riemannian coordinates in 3-space. From these it can be shown (see reference 6, appendix B) that

$$(\Gamma_{\iota,\,\varkappa\lambda}),\,\mu = (\Gamma_{\mu,\,\varkappa\lambda}),\,\iota.\tag{54}$$

Hence, from (50) and (54) one has

$$\hat{t}_{\iota}^{\varkappa} = 0.$$

Thus all the components of the complex $\hat{t}_{i}^{k}$ vanish along the geodesic.

Where $R = 0$, i. e. where no matter is present, or only an electromagnetic field, it is therefore possible to introduce along a geodesic coordinates such that the energy-momentum complex of the gravitational field vanishes on the geodesic.

Acknowledgements

Appendix

In this appendix it will be shown that for a geodesic in an affinely connected space A_N it is possible to introduce coordinate systems such that

$$\begin{aligned} \Gamma^i_{kl} &= 0 & i,\, k,\, l &= 1,\, 2,\ldots,\, N \\ 3\, \underset{(\varkappa\lambda\mu)}{S}\, \Gamma^i_{\varkappa\lambda,\,\mu} &= \Gamma^i_{\varkappa\lambda,\,\mu} + \Gamma^i_{\mu\varkappa,\,\lambda} + \Gamma^i_{\lambda\mu,\,\varkappa} = 0, & \varkappa,\, \lambda,\, \mu &= 1,\, 2,\ldots,\, N-1 \end{aligned}\right\} \quad (A\,1)$$

etc.

at every point of the curve. A complete proof for a general curve in A_N has been given by SCHOUTEN[11]. The present proof is a simplified version of his. Of course the results hold *a fortiori* for a Riemannian space V_N.

Let the equation of the geodesic be given by

$$\xi^i = f^i(l), \quad \xi^i_0 = f^i(0). \tag{A2}$$

Consider the hypersurface orthogonal to the geodesic at the point P_0 with coordinates ξ^i_0. For a neighbouring point Q_0 in this surface, with coordinates ξ^i, there is a unique geodesic passing through P_0 and Q_0. Its direction at P_0 is given by the vector

$$t^i_0 = \left(\frac{d\xi^i}{dz}\right)_0, \tag{A3}$$

where z is an affine parameter on the geodesic. In a Riemannian space, z can always be taken as the arc length along the (non-null) geodesic.

The equation of the geodesic through P_0 and Q_0 is

$$\frac{d^2\xi^i}{dz^2} + \Gamma^i_{kl}\frac{d\xi^k}{dz}\frac{d\xi^l}{dz} = 0. \tag{A4}$$

One may expand the coordinates ξ^i of Q_0 in a series as follows, putting $z = 0$ at P_0:

$$\xi^i = \xi_0^i + \left(\frac{d\xi^i}{dz}\right)_0 z + \frac{1}{2!}\left(\frac{d^2\xi^i}{dz^2}\right)_0 z^2 + \frac{1}{3!}\left(\frac{d^3\xi^i}{dz^3}\right)_0 z^3 + \cdots. \tag{A5}$$

Differentiation of (A4) gives

$$\begin{aligned}
\frac{d^3\xi^i}{dz^3} &= -\left[(\Gamma_{kl}^i)_{,\,m}\frac{d\xi^k}{dz}\frac{d\xi^l}{dz}\frac{d\xi^m}{dz} + 2\Gamma_{kl}^i\frac{d^2\xi^k}{dz^2}\frac{d\xi^l}{dz}\right] \\
&= -\left[(\Gamma_{kl}^i)_{,\,m} - 2\Gamma_{kl}^r\Gamma_{mr}^i\right]\frac{d\xi^k}{dz}\frac{d\xi^l}{dz}\frac{d\xi^m}{dz} \\
&= -\Gamma_{klm}^i\frac{d\xi^k}{dz}\frac{d\xi^l}{dz}\frac{d\xi^m}{dz},
\end{aligned} \tag{A6}$$

where

$$\Gamma_{klm}^i = \underset{(klm)}{S}\left[(\Gamma_{kl}^i)_{,\,m} - 2\Gamma_{kl}^r\Gamma_{mr}^i\right]. \tag{A7}$$

S is a symmetrizing operator defined by

$$\underset{(klm)}{S}\,P_{klm} = \frac{1}{3!}\left(P_{klm} + P_{mkl} + P_{lmk} + P_{mlk} + P_{kml} + P_{lkm}\right). \tag{A8}$$

In general, $\underset{(n_1 n_2 \ldots n_p)}{S}\,P_{n_1 n_2 \ldots n_p}$ is the sum of all $p!$ quantities $P_{n_1 n_2 \ldots n_p}$ with all permutations of $n_1 n_2 \ldots n_p$, divided by $p!$.

Then one finds that

$$\frac{d^p\xi^i}{dz^p} = -\Gamma_{k_1 k_2 \ldots k_p}^i\frac{d\xi^{k_1}}{dz}\frac{d\xi^{k_2}}{dz}\cdots\frac{d\xi^{k_p}}{dz}, \tag{A9}$$

where

$$\Gamma_{k_1 k_2 \ldots k_p}^i = \underset{(k_1 k_2 \ldots k_p)}{S}\left[(\Gamma_{k_1 k_2 \ldots k_{p-1}}^i)_{,\,k_p} - (p-1)\Gamma_{k_1 k_2}^r\Gamma_{k_3 \ldots k_p r}^i\right]. \tag{A10}$$

The coordinates ξ^i of any point Q_0 in the neighbourhood of the geodesic can then, by (A5), (A9) and (A3), be expressed by the equation

$$\begin{aligned}
\xi^i = \xi_0^i &+ t_0^i z - \frac{1}{2!}\Gamma_{kl}^i(\xi_0^i)\,t_0^k t_0^l z^2 - \frac{1}{3!}\Gamma_{klm}^i(\xi_0^i)\,t_0^k t_0^l t_0^m z^3 \\
&- \frac{1}{4!}\Gamma_{klmn}^i(\xi_0^i)\,t_0^k t_0^l t_0^m t_0^n z^4 - \cdots.
\end{aligned} \tag{A11}$$

For every point Q_0 there is one vector t_0^i orthogonal to the geodesic. If one makes a parallel displacement along the geodesic from P_0 to a general point P, with coordinates $\xi^i = f^i(t)$, the vectors t_0^i at P_0 go over into vectors t^i at P orthogonal to the geodesic. The vector t^i will depend on the parameter t of the geodesic and the parameters specifying the vector t_0^i. To every

neighbouring point Q in the orthogonal hypersurface at P corresponds one vector t^i giving the direction at P of the geodesic from P to Q. The coordinates ξ^i of the point Q can then be expressed by an equation corresponding to (A11):

$$\left.\begin{aligned}
\xi^i = f^i(t) + t^i z - \frac{1}{2!}\, \Gamma^i_{kl}\left\{f^i(t)\right\} t^k t^l z^2 - \frac{1}{3!}\, \Gamma^i_{klm}\left\{f^i(t)\right\} t^k t^l t^m z^3 \\
- \frac{1}{4!}\, \Gamma^i_{klmn}\left\{f^i(t)\right\} t^k t^l t^m t^n z^4 - \ldots
\end{aligned}\right\} \quad \text{(A12)}$$

Now at P_0 introduce N linearly independent vectors $e_1^{0\,i}, e_2^{0\,i}, \ldots, e_N^{0\,i}$ such that $e_N^{0\,i}$ is tangential to the geodesic and $e_\mu^{0\,i}$, $\mu = 1, 2, \ldots, N-1$ are orthogonal to it. The vectors $e_\mu^{0\,i}$ span the orthogonal hypersurface at P_0 so that the vectors t_0^i can be expressed in terms of them:

$$t_0^i = t^\mu e_\mu^{0\,i}, \quad \mu = 1, 2, \ldots, N-1. \quad \text{(A13)}$$

The t^μ will depend on the parameters specifying the vector t_0^i. If one makes a parallel displacement along the geodesic from P_0 to P, the vectors $e_\mu^{0\,i}, e_N^{0\,i}$ go over into N linearly independent vectors e_μ^i, e_N^i such that e_N^i is tangential to the geodesic and the e_μ^i orthogonal to it. Thus the e_μ^i span the orthogonal hypersurface at P, and the t^i can be expressed in terms of them:

$$t^i = t^\mu e_\mu^i. \quad \text{(A14)}$$

Unlike the vectors e_μ^i, the t^μ are independent of t, depending only on the parameters specifying the original t_0^i. This is due to the fact that the covariant derivatives of t^i and e_μ^i vanish along the curve.

The coordinates of a point in the neighbourhood of the geodesic may then be expressed by the equation

$$\left.\begin{aligned}
\xi^i = f^i(t) + z t^\mu e_\mu^i(t) - \frac{1}{2!}\, \Gamma^i_{kl}\{t\}\, z^2 t^\mu t^\nu e_\mu^k(t) e_\nu^l(t) \\
- \frac{1}{3!}\, \Gamma^i_{klm}\{t\}\, z^3 t^\mu t^\nu t^\varrho e_\mu^k(t) e_\nu^l(t) e_\varrho^m(t) \\
- \frac{1}{4!}\, \Gamma^i_{klmn}\{t\}\, z^4 t^\mu t^\nu t^\varrho t^\sigma e_\mu^k(t) e_\nu^l(t) e_\varrho^m(t) e_\sigma^n(t) - \ldots
\end{aligned}\right\} \quad \text{(A15)}$$

The ξ^i can thus be given in terms of the N independent variables $z t^\mu$, t. The e_μ^i are known functions of t, depending on the initial choice of $e_\mu^{0\,i}$. Along the curve the Γ^i_{kl} are functions of t only.

Introducing a new coordinate system, defined by

$$\begin{aligned}
\eta^\mu &= zt^\mu, \quad \mu = 1, 2, .., N-1, \\
\eta^N &= t,
\end{aligned} \right\} \quad (A\,16)$$

we shall now show that the basis vectors of this system on the geodesic, i. e. the vectors along the coordinate curves or parametric lines, are the very vectors e^i_μ, e^i_N already defined at every point of the curve. The coordinate system (η) is then defined at every point of the geodesic.

Substituting from (A16) in (A15), one has

$$\begin{aligned}
\xi^i = f^i(\eta^N) + \eta^\mu e^i_\mu(\eta^N) &- \frac{1}{2!}\, \Gamma^i_{kl}\{\eta^N\}\, \eta^\mu\, \eta^\nu\, e^k_\mu(\eta^N)\, e^l_\nu(\eta^N) \\
&- \frac{1}{3!}\, \Gamma^i_{klm}\{\eta^N\}\, \eta^\mu\, \eta^\nu\, \eta^\varrho\, e^k_\mu(\eta^N)\, e^l_\nu(\eta^N)\, e^m_\varrho(\eta^N) \\
&- \frac{1}{4!}\, \Gamma^i_{klmn}\{\eta^N\}\, \eta^\mu\, \eta^\nu\, \eta^\varrho\, \eta^\sigma\, e^k_\mu(\eta^N)\, e^l_\nu(\eta^N)\, e^m_\varrho(\eta^N)\, e^n_\sigma(\eta^N) - \cdots.
\end{aligned} \right\} \quad (A\,17)$$

The equation of the geodesic in the new coordinate system is

$$\eta^\mu = 0, \quad \mu = 1, 2, .., N-1. \quad (A\,18)$$

The kth basis vector of the new system, i. e. the vector along the kth coordinate curve $\eta^i = $ const., $i \neq k$, is $\dfrac{\partial \xi^i}{\partial \eta^k}$ in the old coordinate system since $d\xi^i = \dfrac{\partial \xi^i}{\partial \eta^k}\, d\eta^k$. On the geodesic one finds from (A17)

$$\left(\frac{\partial \xi^i}{\partial \eta^\mu}\right)_{\eta^\mu = 0} = e^i_\mu(\eta^N), \quad \left(\frac{\partial \xi^i}{\partial \eta^N}\right)_{\eta^\mu = 0} = \frac{df^i}{dt} = e^i_N, \quad (A\,19)$$

which was to be shown.

Equation (A17) expresses the general coordinates ξ^i in terms of the particular coordinates η^k. This equation holds for any coordinates ξ^i, in particular for $\xi^i = \eta^i$. In that case one has $e^i_k = \dfrac{\partial \eta^i}{\partial \eta^k} = \delta^i_k$ so that equation (A17) becomes

$$\begin{aligned}
\eta^i = f^i(\eta^N) + \eta^\mu \delta^i_\mu &- \frac{1}{2!}\, \Gamma^i_{\mu\nu}\{\eta^N\}\, \eta^\mu\, \eta^\nu \\
&- \frac{1}{3!}\, \Gamma^i_{\mu\nu\varrho}\{\eta^N\}\, \eta^\mu\, \eta^\nu\, \eta^\varrho - \frac{1}{4!}\, \Gamma^i_{\mu\nu\varrho\sigma}\{\eta^N\}\, \eta^\mu\, \eta^\nu\, \eta^\varrho\, \eta^\sigma - \cdots
\end{aligned} \right\} \quad (A\,20)$$

Differentiating this equation with respect to (η^μ, η^ν), $(\eta^\mu, \eta^\nu, \eta^\varrho)$, $(\eta^\mu, \eta^\nu, \eta^\varrho, \eta^\sigma)$, etc., and putting $\eta^\mu = 0$, one finds that on the geodesic

$$
\left.
\begin{aligned}
\Gamma^i_{\mu\nu}\{\eta^N\} &= 0 \\
\Gamma^i_{\mu\nu\varrho}\{\eta^N\} &= 0 \\
\Gamma^i_{\mu\nu\varrho\sigma}\{\eta^N\} &= 0, \\
&\text{etc.}
\end{aligned}
\right\} \quad \text{(A 21)}
$$

Along the curve $\xi^i = f^i(t)$ the covariant derivatives of the vectors e^i_k vanish ($k = \mu$, N is not a vector index, but a label for the different vectors), by definition, i. e.

$$
\frac{de^i_k}{dt} + \Gamma^i_{rs}\{t\}\, e^r_k\, \frac{df^s}{dt} = 0 \qquad \text{(A 22)}
$$

or, by equation (A 19),

$$
\frac{de^i_k}{dt} + \Gamma^i_{rs}\{t\}\, e^r_k\, e^s_N = 0. \qquad \text{(A 23)}
$$

In the coordinate system (η) one has $e^i_k = \delta^i_k$. Therefore, in that system (A 23) shows that on the curve

$$
\Gamma^i_{kN}\{\eta^N\} = 0, \quad k = 1, 2, \ldots, N. \qquad \text{(A 24)}
$$

Now, since $\Gamma^i_{\mu\nu} = 0$, one has

$$
\Gamma^i_{\mu\nu\varrho} = \mathop{S}_{(\mu\nu\varrho)} [\Gamma^i_{\mu\nu,\,\varrho} - 2\,\Gamma^r_{\mu\nu}\,\Gamma^i_{\varrho r}] = \mathop{S}_{(\mu\nu\varrho)} \Gamma^i_{\mu\nu,\,\varrho} \qquad \text{(A 25)}
$$

and

$$
\Gamma^i_{\mu\nu\varrho\sigma} = \mathop{S}_{(\mu\nu\varrho\sigma)} [\Gamma^i_{\mu\nu\varrho,\,\sigma} - 3\,\Gamma^r_{\mu\nu}\,\Gamma^i_{\varrho\sigma r}] = \mathop{S}_{(\mu\nu\varrho\sigma)} \Gamma^i_{\mu\nu\varrho,\,\sigma} = \mathop{S}_{(\mu\nu\varrho\sigma)} \Gamma^i_{\mu\nu,\,\varrho,\,\sigma}. \qquad \text{(A 26)}
$$

From (A 21) and (A 24) — (A 26) it is then found that

$$
\left.
\begin{aligned}
\Gamma^i_{kl} &= 0 && i, k, l = 1, 2, \ldots, N \\
\mathop{S}_{(\mu\nu\varrho)} \Gamma^i_{\mu\nu,\,\varrho} &= 0 && \mu, \nu, \ldots = 1, 2, \ldots, N-1 \\
\mathop{S}_{(\mu\nu\varrho\sigma)} \Gamma^i_{\mu\nu,\,\varrho,\,\sigma} &= 0, \\
&\text{etc.}
\end{aligned}
\right\} \quad \text{(A 27)}
$$

at all points of the curve. The coordinates for which the equations (A 27) hold are called Fermi coordinates by SCHOUTEN, for Fermi was the first to show that coordinates can be chosen along any curve in Riemannian space V_N so that $\Gamma^i_{kl} = 0$ at every point of the curve.

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